365!

David Beckwith, February 2024

One of my favourite Oxbridge questions was: - *How many trailing zeros does 365! have when it is expanded?* For example: - 13! = 6227020800 which has 2 zeros at the end. I used it as an early problem for potential Oxbridge mathematicians and as a discussion problem for a practice Oxbridge maths interview. Try for yourself – or perhaps just see the solution on the last page of this article.

Recently, I was looking at Which Way did the Bicycle Go?, by Konhauser, Velleman & Wagon. This wonderful book has nearly 200 challenging but interesting problems and their solutions. The title of the book is another of my favourite problems! Thank you for the gift Russell.

Problem 90 asks: - *What is the rightmost non-zero digit of 1,000,000!* i.e. the digit just before the trailing zeros.
It reminded me of the 365! problem above, and it would make a fine follow up question using 365!
I thought that I would try to solve it . . .

I had a couple of ideas which didn't work (i.e. I failed!), so I shall use the book's method.

Firstly, if we want to find the last digit of the product ab, then we need only consider the last digits of a and b.
For example: the last digit of $1234 \cdot 9876 = 12{,}186{,}984$ is the same as the last digit of $4 \cdot 6 = 24$, i.e. 4.
This is easily seen by considering long multiplication!

It also helps when looking for the last digit of a power. For example: the last digit of $2^{20} = 2^{10} \cdot 2^{10} = 1024 \cdot 1024$ is the same as the last digit of $4 \cdot 4 = 16$ which is 6.

When looking for the rightmost non-zero digit, we need to be more careful.
We shall use a wavy equals sign to denote that a pair of numbers have the same rightmost non-zero digit.
For example: $123 \approx 54300$ and $2^{11} \approx 8$.
When multiplying numbers, things are tricky if extra trailing zeros arise.
For example: $2 \approx 12$ but multiplying by five gives 10 and 60, with different rightmost non-zero digits.
However, multiplying by 10 is fine, as is multiplying with any non-multiple of 5. This leads to . . .

Lemma 1 :

If 5 is not the rightmost non-zero digit of a, b, or c, then $a \approx b \ \Rightarrow ac \approx bc$

In maths, a lemma is a like a small theorem. They appear as preliminary results used in the proof of a main theorem or solution of the main problem.

We shall also need a less obvious lemma which we need to prove . . .

Lemma 2 :

$$(5n)! \approx 2^n \cdot n!$$

Proof:

$$\begin{aligned}
(5n)! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20 \cdots 5n \\
&= (1 \cdot 2 \cdot 3 \cdot 4)(6 \cdot 7 \cdot 8 \cdot 9)(11 \cdot 12 \cdot 13 \cdot 14)(16 \cdot 17 \cdot 18 \cdot 19) \cdots \{5 \cdot 10 \cdot 15 \cdot 20 \cdots 5n\} \\
&= (1 \cdot 2 \cdot 3 \cdot 4)(6 \cdot 7 \cdot 8 \cdot 9)(11 \cdot 12 \cdot 13 \cdot 14)(16 \cdot 17 \cdot 18 \cdot 19) \cdots 5^n \cdot \{1 \cdot 2 \cdot 3 \cdot 4 \cdots n\} \\
&= (1 \cdot 1 \cdot 3 \cdot 4)(3 \cdot 7 \cdot 8 \cdot 9)(11 \cdot 6 \cdot 13 \cdot 14)(8 \cdot 17 \cdot 18 \cdot 19) \cdots 10^n \cdot n!
\end{aligned}$$

having taken a factor or 2 out of each pair of round brackets.

$$\approx (1 \cdot 1 \cdot 3 \cdot 4)(3 \cdot 7 \cdot 8 \cdot 9)(11 \cdot 11 \cdot 13 \cdot 14)(13 \cdot 17 \cdot 18 \cdot 19) \cdots n! \quad \text{ignoring the trailing zeros.}$$

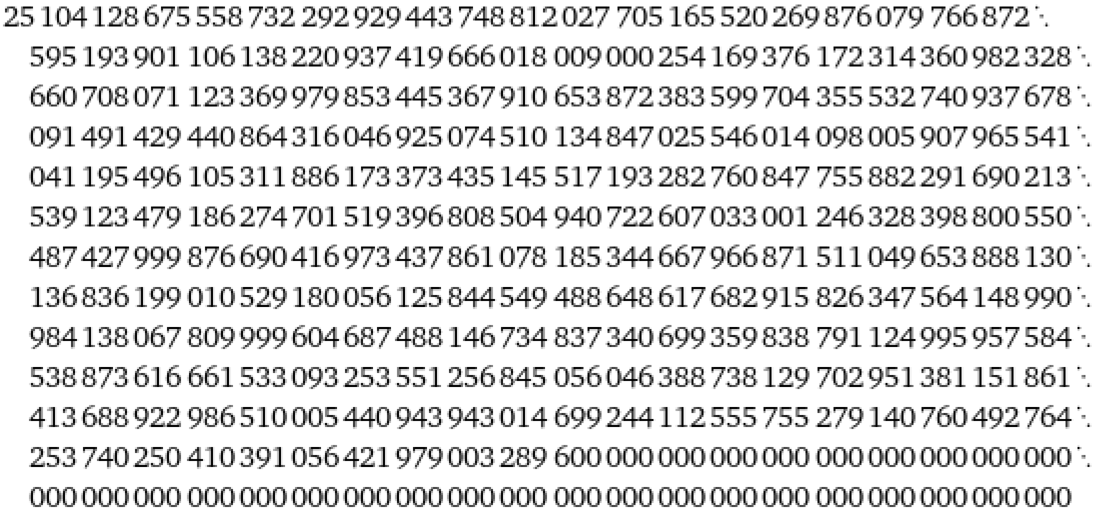
Now consider the terms in the round brackets at line 3, where there are no numbers ending in a 5.
$1 \cdot 2 \cdot 3 \cdot 4 \approx 11 \cdot 12 \cdot 13 \cdot 14 \approx 21 \cdot 22 \cdot 23 \cdot 24 \approx . . . \approx 4$, since we are only considering the last digit, and
$6 \cdot 7 \cdot 8 \cdot 9 \approx 16 \cdot 17 \cdot 18 \cdot 19 \approx 26 \cdot 27 \cdot 28 \cdot 29 \approx . . . \approx 4$ also.
We shall now take out a factor of 2 from all n terms, leaving each with a value of 2.
Also, n! cannot end in a 5, so Lemma 1 applies.

Hence, $(5n)! \approx 2^n \cdot n!$ Q.E.D.

We can now apply this Lemma 2 recursively . . .

$$\begin{aligned}
365! &\approx 2^{73} \cdot 73! \\
&= 2^{73} \cdot 70! \cdot 71 \cdot 72 \cdot 73 \\
&\approx 2^{73} \cdot 70! \cdot 6 \\
&\approx 2^{73} \cdot 2^{14} \cdot 14! \cdot 6 \\
&= 2^{87} \cdot 10! \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 6 \\
&\approx 2^{87} \cdot 10! \cdot 4 \\
&\approx 2^{87} \cdot 2^{2} \cdot 2! \cdot 4 \\
&= 2^{92} \\
&= 16^{23} \\
&\approx 6 \text{ , since powers of a number ending in 6 always ends in 6.}
\end{aligned}$$

Wolfram Alpha give the exact value as: -



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25 104 128 675 558 732 292 929 443 748 812 027 705 165 520 269 876 079 766 872 ⋱
  595 193 901 106 138 220 937 419 666 018 009 000 254 169 376 172 314 360 982 328 ⋱
  660 708 071 123 369 979 853 445 367 910 653 872 383 599 704 355 532 740 937 678 ⋱
  091 491 429 440 864 316 046 925 074 510 134 847 025 546 014 098 005 907 965 541 ⋱
  041 195 496 105 311 886 173 373 435 145 517 193 282 760 847 755 882 291 690 213 ⋱
  539 123 479 186 274 701 519 396 808 504 940 722 607 033 001 246 328 398 800 550 ⋱
  487 427 999 876 690 416 973 437 861 078 185 344 667 966 871 511 049 653 888 130 ⋱
  136 836 199 010 529 180 056 125 844 549 488 648 617 682 915 826 347 564 148 990 ⋱
  984 138 067 809 999 604 687 488 146 734 837 340 699 359 838 791 124 995 957 584 ⋱
  538 873 616 661 533 093 253 551 256 845 056 046 388 738 129 702 951 381 151 861 ⋱
  413 688 922 986 510 005 440 943 943 014 699 244 112 555 755 279 140 760 492 764 ⋱
  253 740 250 410 391 056 421 979 003 289 600 000 000 000 000 000 000 000 000 000 ⋱
  000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000
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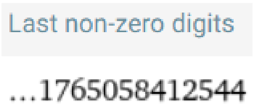
Yay! Its rightmost non-zero digit is indeed 6. 😀

The book Problem 90 asked for the rightmost non-zero digit of 1,000,000!
This is too big for Wolfram Alpha to display in the online version, as it has 5,565,709 digits.
Wolfram Mathematica could expand it exactly if it was given enough memory and time!

Alpha does give:



Last non-zero digits

...1765058412544

Its rightmost non-zero digit is 4, as found in the book.
I wonder how Alpha worked out the other digits?!?

Spoiler alert . . . my solution to the 365! Oxbridge question is given on the next page.

How many trailing zeros does 365! have when it is expanded?

Trailing zeros are generated when a multiple of 5 is multiplied by an even number.

There are 73 multiples of 5 in the list 1,2,3,4,5,6, . . . ,365 and many more even numbers.

However, 73 is not the correct answer! (A common error by my students.) There are more trailing zero . . .

This is because a multiple of 25 contribute a second factors of 5, and there are 14 of them.

Further, 125 multiples each contribute a third factor of 5, which give 2 more.

We have a total or 89 factors of 5 in the prime factorisation of 365! And there are many more factors of 2.

Hence, 365! has 89 trailing zeros.